Project 8 Memo

For problem 4

Choose the domain \([a_0, a_1] \times [b_0, b_1]\) for different \(a\) values as follows:

(1) \(a = 1\), choose \([a_0, a_1] \times [b_0, b_1] = [-4, -0.1] \times [0, 6.3]\), i.e., \(t_1 \in [-4, -0.1]\) and \(t_2 \in [0, 6.3]\),
(2) \(a = 0.8\), choose \([0.1, 4] \times [0.46, 20]\),
(3) \(a = 0.6\), choose \([-3, 0.1] \times [0.21, 12]\),
(4) \(a = 0.4\), choose \([-3, 0.1] \times [0.13, 12]\),
(5) \(a = 0.2\), choose \([-4, 0.1] \times [0.09, 8]\).

If the domain is \([a_0, b_0] \times [a_1, b_1]\), then we choose the initial data as follows: the initial point, and initial frame \(v_1, v_2, v_3\):

\[
\begin{align*}
X(a_0, b_0) &= (0, 0, 0), \\
v_1(a_0, b_0) &= (\sqrt{g_{11}(a_0, b_0)}, 0, 0), \\
v_2(a_0, b_0) &= (0, \sqrt{g_{22}(a_0, b_0)}, 0), \\
v_3(a_0, b_0) &= (0, 0, 1)
\end{align*}
\]

Look at the formula of the first fundamental form, we see that \(g_{11}g_{22} = 0\) exactly when the function

\[C(t_1, t_2) = 1,\]

i.e., when

\[(a + a^{-1})t_1 + (a - a^{-1})t_2 = 0,
\]

or equivalently

\[\frac{t_2}{t_1} = \frac{a + a^{-1}}{a^{-1} - a}.
\]

So if you choose any rectangular region domain that does not meet this line, then \(g_{11}g_{22}\) will stay positive and your program should work.

For Problem 5

You need to choose your rectangular domain away from where \(B(t_1, t_2) = \pm 1\), which can not be solved explicitly. So it will be difficult to get a good rectangular region for the domain. However, there is another way:

An alternative better method: use orthonormal frame

We have seen in the last lecture that for surfaces with \(g_{12} = \ell_{12} = 0\), the Gauss-Codazzi equation has only three equations. Moreover, if you modify your program using the orthonormal frame, then even when \(g_{11}g_{22} = 0\) somewhere, you will be able to solve the Gauss-Codazzi
equation. To be more specific, write a program to solve the following system

\[
\begin{align*}
(e_1, e_2, e_3)_{x_1} &= (e_1, e_2, e_3) \begin{pmatrix} 0 & -q_{x_2} & -\sin q \\ q_{x_2} & 0 & 0 \\ \sin q & 0 & 0 \end{pmatrix}, \\
(e_1, e_2, e_3)_{x_2} &= (e_1, e_2, e_3) \begin{pmatrix} 0 & -q_{x_1} & 0 \\ q_{x_1} & 0 & \cos q \\ 0 & -\cos q & 0 \end{pmatrix},
\end{align*}
\]

(0.0.1)

Then solve

\[
\begin{align*}
X_{x_1} &= \cos q \ e_1, \\
X_{x_2} &= \sin q \ e_2.
\end{align*}
\]

(0.0.2)

In this case, you can choose any domain and initial data and the program won’t break down.

Note that if \( q = 2 \tan^{-1} B \), then

\[
q_{x_i} = \frac{2B_x}{1+B^2}, \quad \cos q = \frac{1-B^2}{1+B^2}, \quad \sin q = \frac{2B}{1+B^2}.
\]

If you have finished the final Matlab project 8 for general coordinates, then you can modify a little bit what you have, to write a program for the above simpler systems (0.0.1), (0.0.2), and the resulting \( X \) is your surface.

Use this new program to do problem 5, you may use initial data

\[
X(a_0, b_0) = (0, 0, 0),
\]

\[
e_1(a_0, b_0) = (1, 0, 0), \quad e_2(a_0, b_0) = (0, 1, 0), \quad e_3(a_0, b_0) = (0, 0, 1),
\]

and the domain \([a_0, a_1] \times [b_0, b_1]\) can be chosen to be

1. \([-8, 8] \times [-10, 10]\) if \( \alpha = 0.6 \),
2. \([-8, 8] \times [-16, 16]\) if \( \alpha = 0.8 \).