Second Assignment Answers

Some exercises on linear transformations and matrices.

▷ Exercise 1. Let \( v_1, \ldots, v_n \) be a basis for a vector space \( V \) and let \( S \) and \( T \) be two linear operators on \( V \). If the matrices of \( S \) and \( T \) relative to this basis are respectively \( S_{ij} \) and \( T_{ij} \), then show that the matrix elements of the composed linear operator \( ST \) are given by \( (ST)_{ij} = \sum_{k=1}^{n} S_{ik}T_{kj} \), and that the matrix elements of the sum operator \( S + T \) are given by \( (S + T)_{ij} = S_{ij} + T_{ij} \).

Answer  

By definition of \( S_{ij} \) and \( T_{ij} \) we have:  
\[
Tv_j = \sum_{i=1}^{n} T_{ij}v_i  
\]
and similarly  
\[
Sv_j = \sum_{i=1}^{n} S_{ij}v_i.
\]
Since (by definition of the addition of linear operators)  
\[
(S + T)(v_j) = S(v_j) + T(v_j),
\]
the formula for \( (S + T)_{ij} \) is immediate. On the other hand the “product” \( ST \) is defined to be the composition of \( S \) and \( T \), so  
\[
(ST)(v_j) = S(T(v_j)) = S(\sum_{i=1}^{n} T_{ij}v_i) = \sum_{i=1}^{n} T_{ij}S(v_i) = \sum_{i=1}^{n} T_{ij} \sum_{k=1}^{n} S_{ki}v_k = \sum_{k=1}^{n} (ST)_{kj}v_k.
\]

In what follows, \( \mathcal{P}^n \) denotes the space of polynomials functions \( a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \) of degree \( \leq n \). Clearly \( \mathcal{P}^n \) is a vector space of dimension \( n + 1 \) and \( 1, x, x^2, \ldots, x^n \) is a basis for \( \mathcal{P}^n \) (called the standard basis).

▷ Exercise 2. Differentiation defines an operator \( D \) on \( \mathcal{P}^n \), and of course \( D^k \) denotes the \( k \)-th derivative operator.

a) What is the matrix of \( D \) in the standard basis?

Answer  

Let’s denote the standard basis by \( v_0 = 1, v_1 = x, \ldots, v_n = x^n \). Then since  
\[
Dv_k = kv_{k-1},
\]
the matrix \( D_{ij} \) of \( D \) is given by \( D_{ij} = j \) if \( i = j - 1 \) and \( D_{ij} = 0 \) otherwise. (Note that this says that the first or “row-index” must be one less than the second or “column index” for a matrix element to be non-zero.) So the non-zero entries are  
\( D_{01} = 1, D_{12} = 2, \ldots, D_{n-1,n} = n \), i.e., the matrix has 1, 2, \ldots, \( n \) just above the diagonal and zero elsewhere.

b) What is the kernel of \( D^k \)?

Answer \( \mathcal{P}^{k-1} \)

c) What is the image of \( D^k \)?

Answer \( \mathcal{P}^{n-k} \)

▷ Exercise 3. Define an inner-product on \( \mathcal{P}^n \) by  
\[
\langle P_1, P_2 \rangle = \int_{-1}^{1} P_1(x)P_2(x) \, dx,
\]
and note that the standard basis is not orthonormal (or even orthogonal). Let us define orthonormal polynomials \( L_k(x) \) by applying the Gram-Schmidt Algorithm to the standard basis. (The \( L_k \) are usually called the normalized Legendre Polynomials.) Compute \( L_0, L_1, \) and \( L_2 \).

Answer  

\( L_0 = \sqrt{2} \), \( L_1 = \frac{\sqrt{6}}{2} x \), \( L_2 = \frac{\sqrt{10}}{2} (3x^2 - 1) \).